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Steady-state heat transfer from slab-on-grade floors with vertical insulation

MONCEF KRARTI
Joint Center for Energy Management, CEAE Department,
University of Colorado, Boulder, CO 80309-0428, U.S.A.

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Abstract—The steady-state temperature field distribution beneath a vertically insulated slab is derived using the Interzone Temperature Profile Estimation (ITPE) technique. A water table is considered at a finite depth below the soil surface. The heat flux variation along the slab is discussed as well as the effect of vertical insulation length and value on the total heat losses from the slab-on-grade floor. It is shown that when the depth and/or the thermal resistance of the vertical insulation increases, slab heat loss decreases following the law of diminishing returns. The water table level was found to have a significant effect on total slab heat loss.

1. INTRODUCTION

Two approaches to insulating slab-on-grade foundations are commonly used: (1) horizontal insulation under or above the slab, and (2) vertical insulation along the foundation walls. The horizontal insulation is usually placed under the slab perimeter. However, the insulation can be placed above the floor slab with the option of using batt insulation rather than rigid insulation board. This approach reduces construction problems inherent to the more conventional approaches such as the need of insulating the joint between the slab and the foundation wall. This joint, if kept uninsulated, may lead to a significant amount of heat transfer between building interior and earth and/or ambient air. In cold climates, horizontal insulation configurations can increase the frost depth in the vicinity of the slab edge and thus may cause structural damages to the foundation walls. To avoid frost heaving, an effective approach is vertical insulation.

Two options are recommended for vertical insulation, either on the exterior or the interior of the foundation wall. The exterior insulation is easy to install but needs a protective coating; the interior insulation does not need protection but is more difficult to install since the joint between the slab and the foundation wall also needs to be insulated, and this approach is not suitable for beam-on-grade construction used in some southern states.

Several models treating the problem of heat losses from slab-on-grade floors have been developed [1, 2]. However, most of these models deal with heat losses from horizontally insulated slabs. In particular, analytical models for heat transfer from slabs with partial, horizontal insulation have been developed [3–5]. Models for slabs with vertical insulation are very few and are restricted to limited configurations. Landman and Delsante [6] have developed a Fourier series solution for a vertically insulated slab. Their model does not allow for existence of a constant-temperature water table below the slab. Shen [7] has extended his finite difference model to include vertical insulation configurations. This model was used to generate data presented in the Building Foundation Design Handbook [8]. Unfortunately, the reported results are restricted to certain insulation lengths and values.

In this paper, the steady-state temperature field distribution beneath vertically insulated slabs is analyzed. The heat conduction solution beneath a vertically insulated slab-on-grade floor is derived using the ITPE method [9–12]. This method combines analytical and numerical techniques to arrive at the functional form of the solution and allows detailed analysis of the heat transfer interaction between the slab and the ground. A water table is considered at a finite depth below the soil surface. The heat flux variation along the slab is discussed as well as the effect of vertical insulation length and U-value on the total heat losses from the slab-on-grade floor.

2. VERTICALLY INSULATED SLAB-ON-GRADE FLOOR MODEL

Figure 1 shows a model of a vertically insulated slab-on-grade floor. The vertical insulation is placed along the foundation walls. These walls are usually made of concrete and their thermal conductivity is, in most cases, similar to that of the ground. Therefore, the foundation walls and footings are considered as integral parts of the ground medium.

The steady-state temperature distribution $T(x, y)$ inside the ground beneath the vertically insulated slab-on-grade floor model of Fig. 1 is subject to Laplace equation:

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \qquad (1)$$

NOMENCLATURE

a	building half width [m]
b	water table depth [m]
c	length of vertical insulation [m]
C_n, f_n, g_n	Fourier coefficients
H_f	ratio, h_f/k_s [m^{-1}]
H_v	ratio, h_v/k_s [m^{-1}]
h_f	slab overall heat transfer conductance [$W m^{-2} K^{-1}$]
h_i	contact conductance for the interface slab–earth [$W m^{-2} K^{-1}$]
h_o	convective heat transfer coefficient above the slab [$W m^{-2} K^{-1}$]
h_v	vertical insulation conductance [$W m^{-2} K^{-1}$]
k_s	soil thermal conductivity [$W m^{-1} K^{-1}$]
T	soil temperature [K]
T_i	building air temperature [K]
T_s	soil surface temperature [K]
T_w	water table temperature [K]
U_i	insulation conductance [$W m^{-2} K^{-1}$]
U_s	slab material conductance [$W m^{-2} K^{-1}$]
x, y	space coordinates [m].

Greek symbols

$\alpha_p^c, \beta_{n,p}^c$	coefficients defined in equation (5)
$\alpha_p^f, \beta_{n,p}^f$	coefficients defined in equation (11)
$\alpha_p^g, \beta_{n,p}^g \gamma_p^g$	coefficients defined in equation (7)
μ_n, ν_n	eigenvalues [m^{-1}]
θ	step function introduced in equation (10).

Subscripts

f	building floor slab
v	vertical insulation
I, II	zone (I), zone (II).

with the boundary conditions:

$$T = T_w \qquad \text{for } y = b$$

$$T = T_s \qquad \text{for } y = 0 \text{ and } |x| > a$$

$$\frac{\partial T}{\partial y} = H_f(T - T_i) \qquad \text{for } y = 0 \text{ and } |x| < a$$

and

$$\frac{\partial T}{\partial x} = H_v(T(|a|^-, y) - T(|a|^+, y)) \quad \text{for } |x| = a \quad \text{and } y < c$$

where

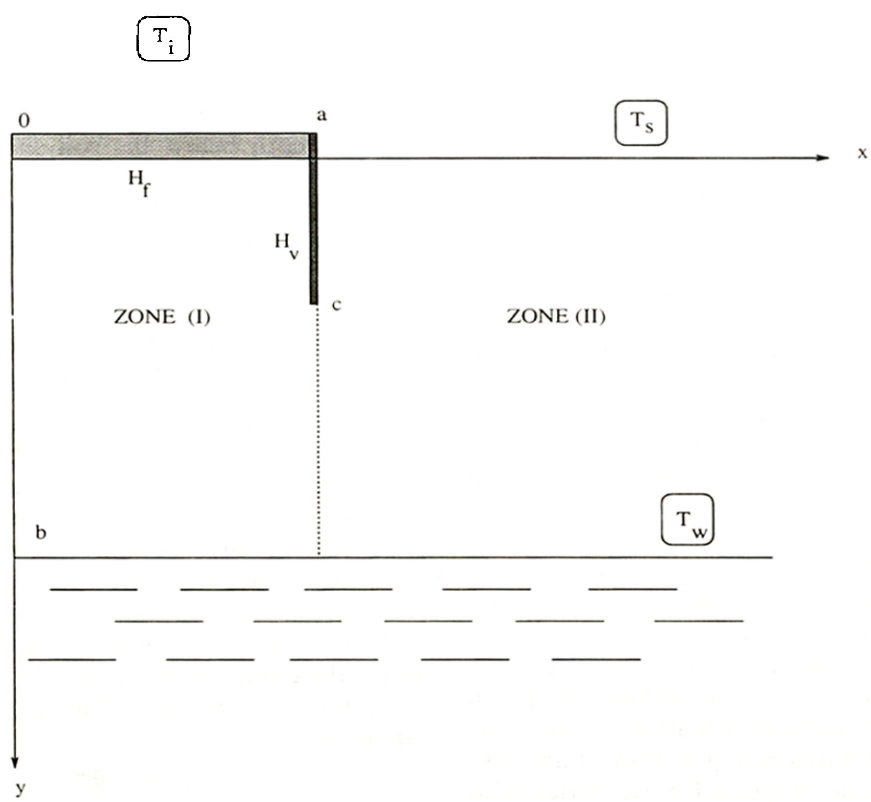


FIG. 1. Slab-on-grade floor with vertical insulation and finite water table level.

- H_f is the ratio of the equivalent air–insulation–slab–soil conductance to the soil thermal conductivity. In other words:

$$H_f = (h_o^{-1}+U_i^{-1}+U_s^{-1}+h_i^{-1})^{-1}/k_s$$

- H_v is the ratio of the vertical insulation U-value to the soil thermal conductivity (i.e. $H_v = U_v/k_s$).
- $T(|a|^-, y)$ and $T(|a|^+, y)$ represent the ground temperatures at the inner and outer side of the vertical insulation, respectively.
- c is the depth of the vertical insulation.

In Fig. 1, the ground medium is divided into three zones by the surfaces $x = -a$ and $x = a$. Because of the symmetry around the axis $x = 0$, only two thermal zones are considered in our analysis: zone (I) just below the slab and zone (II) underneath the soil surface. Let $f(y)$, $g(y)$ be the temperature profiles along the outer and the inner side of the surfaces $|x| = a$, respectively. The solution $T(x, y)$ of equation (1) can be expressed as follows:

In zone (I):

$$T_I(x,y) = \frac{2}{b}\sum_{n=1}^{+\infty} g_n \sin \nu_n y \frac{\cosh \nu_n x}{\cosh \nu_n a} + \frac{2}{a}\sum_{n=1}^{+\infty} C_n \cos \mu_n x \frac{\sinh \mu_n (b-y)}{\sinh \mu_n b}. \quad (2)$$

In zone (II):

$$T_{II}(x,y) = \frac{2}{b}\sum_{n=1}^{+\infty} \sin \nu_n y \left\{\frac{T_s}{\nu_n}(1-e^{-\nu_n(|x|-a)}) + f_n e^{-\nu_n(|x|-a)}\right\} \quad (3)$$

where

$$\nu_n = \frac{n\pi}{b}; \quad \mu_n = \frac{(2n-1)}{2a}\pi$$

and C_n, f_n and g_n are Fourier coefficients to be determined.

The third-kind boundary condition along the slab (i.e. $y = 0$) of equation (1);

$$\left.\frac{\partial T_I}{\partial y}\right|_{y=0} = H_f(T_I - T_i)$$

can be expressed using equation (3) as:

$$\frac{2}{b}\sum_{n=1}^{-\infty} \nu_n g_n \frac{\cosh \nu_n x}{\cosh \nu_n a} - \frac{2}{a}\sum_{n=1}^{+\infty} \mu_n C_n \coth \mu_n b \cos \mu_n x = H_f\left(\frac{2}{a}\sum_{n=1}^{+\infty} C_n \cos \mu_n x - T_i\right). \quad (4)$$

Multiplying this equation by cos $\mu_p x$ and integrating over the interval $[-a, a]$ gives, after rearrangement, an equation of the form:

$$C_p = \alpha_p^c + \sum_{n=1}^{+\infty} \beta_{n,p}^c g_n \quad (5)$$

with

$$\alpha_p^c = -\frac{(-1)^p H_f T_i}{\mu_p(H_f+\mu_p \coth \mu_p b)}$$

$$\beta_{n,p}^c = -\frac{2}{b}\frac{(-1)^p \nu_n \mu_p}{(\mu_p^2+\nu_n^2)(H_f+\mu_p \coth \mu_p b)}.$$

The continuity of the heat flux at the surfaces $|x| = a$, and $0 < y < b$, gives the condition:

$$\left.\frac{\partial T_{II}}{\partial x}\right|_{|x|=a} = \left.\frac{\partial T_I}{\partial x}\right|_{|x|=a} \quad (6)$$

$$\frac{2}{b}\sum_{n=1}^{+\infty}\{T_s - \nu_n f_n\}\sin \nu_n y = \frac{2}{b}\sum_{n=1}^{+\infty} \nu_n g_n \tanh \nu_n a \sin \nu_n y + \frac{2}{a}\sum_{n=1}^{+\infty}(-1)^n \mu_n C_n \frac{\sinh \mu_n (b-y)}{\sinh \mu_n b}.$$

Multiplying the above expression by sin $\nu_p y$ and integrating over $[0, b]$ yields a system of equations:

$$f_p = \alpha_p^g + \sum \beta_{n,p}^g C_n + \gamma_p^g g_p$$

$$\alpha_p^g = \frac{T_s}{\nu_p \tanh \nu_p a}$$

$$\beta_{n,p}^g = -\frac{2}{a}\frac{(-1)^n \mu_n}{(\mu_n^2+\nu_p^2)\tanh \nu_p a}$$

$$\gamma_p^g = \frac{1}{\tanh \nu_p a}. \quad (7)$$

Finally, the third-kind boundary condition along the vertical insulation can be written as

$$\frac{1}{H_v}\left.\frac{\partial T_{II}}{\partial x}\right|_{|x|=a} = (T_{II}-T_I)|_{|x|=a} \quad \text{for } y \leqslant c \quad (8)$$

and the continuity of temperature below the vertical insulation can be expressed as

$$0 = (T_{II}-T_I)|_{|x|=a} \quad \text{for } c < y \leqslant b. \quad (9)$$

Using the expressions of $T_{II}(x, y)$ and $T_I(x, y)$, the above equations (8) and (9) can be combined into:

$$\frac{\theta(y-c)}{H_v}\frac{2}{5}\sum_{n=1}^{+\infty}\{T_s - \nu_n f_n\}\sin \nu_n y = \frac{2}{b}\sum_{n=1}^{+\infty}(f_n - g_n)\sin \nu_n y \quad (10)$$

where $\theta(y-c)$ is a step function defined as

$$\theta(y-c) = \begin{cases} 1 & \text{if } 0 \leqslant y < c \\ 0 & \text{if } c < y \leqslant b \end{cases}.$$

Multiplying equation (10) by sin $\nu_p y$ and integrating over $[0, b]$ gives the following system of equations:

$$f_p = \alpha_p^f + \sum_{n=1}^{+\infty} \beta_{n,p}^f f_n + g_p \quad (11)$$

with

$$\alpha_p^f = \frac{2}{b}\sum_{n=1}^{+\infty}\frac{A_{n,p}}{H_v}$$

$$\beta_{n,p}^f = -\frac{2}{b}\frac{v_n A_{n,p}}{H_v}$$

and

$$A_{n,p} = \frac{1}{2}\left\{\frac{\sin(v_n - v_p)c}{(v_n - v_p)} - \frac{\sin(v_n + v_p)c}{(v_n + v_p)}\right\}.$$

Note that in the case $c = 0$ (no vertical insulation) $A_{n,p} = 0$, and equation (11) reduces to $f_p = g_p$ (i.e. continuity of temperature along the surfaces $|x| = a$). This case represents the uniformly insulated slab configuration treated in detail in ref. [9].

The system of equations (5), (7), and (11) can be solved by truncating the sum to a finite number of terms, N. A linear system of $3N$ equations with $3N$ unknowns (C_p, f_p, and g_p, $p = 1, 2, \ldots, N$) is then obtained. This system is solved using the Gauss–Jordan elimination method. The temperature distributions $T_\text{I}(x, y)$ and $T_\text{II}(x, y)$ are then determined by substituting the values of the coefficients c_n, f_n, and g_n into equations (2) and (3). For the slab configurations considered in this paper, it was found that $N = 30$ gives accurate estimations. Addition of other terms does not affect the results for $T_\text{I}(x, y)$ and $T_\text{II}(x, y)$ significantly (less than 0.05 K variation in soil temperature, at least for the cases treated in this paper).

3. SOIL TEMPERATURE DISTRIBUTION

Figure 2 shows temperature isotherms within the soil below a slab of width $2a = 10$ m. The air above the slab is at temperature $T_i = 21°\text{C}$, while the soil surface is at temperature $T_s = 16°\text{C}$. A water table at a depth $b = 5$ m below the soil surface has a temperature $T_w = 11°\text{C}$. Six different configurations are considered in Fig. 2.

(a) Uninsulated slab without any vertical insulation, ($H_f = 4\ \text{m}^{-1}$; $c = 0$ m), Fig. 2(a).
(b) Uninsulated slab with insulated foundation wall. The vertical insulation extends 0.5 m below the surface, ($H_f = 10\ \text{m}^{-1}$, $H_v = 0.1\ \text{m}^{-1}$, $c = 0.5$ m), Fig. 2(b).
(c) Uninsulated slab with well insulated foundation wall. The vertical insulation extends 0.5 m below the surface, ($H_f = 4\ \text{m}^{-1}$, $H_v = 0.05\ \text{m}^{-1}$, $c = 0.5$ m), Fig. 2(c).
(d) Insulated slab with insulated foundation wall. The vertical insulation extends 0.5 m below the surface, ($H_f = 0.5\ \text{m}^{-1}$, $H_v = 0.1\ \text{m}^{-1}$, $c = 0.5$ m), Fig. 2(d).
(e) Uninsulated slab with insulated foundation wall. The vertical insulation extends 2 m below the surface, ($H_f = 4\ \text{m}^{-1}$, $H_v = 0.1\ \text{m}^{-1}$, $c = 2$ m), Fig. 2(e).
(f) Uninsulated slab with insulated foundation wall. The vertical insulation extends 5 m below the surface, ($H_f = 4\ \text{m}^{-1}$, $H_v = 0.1\ \text{m}^{-1}$, $c = 5$ m), Fig. 2(f).

The effect of adding vertical insulation at the edge of a slab on soil temperature distribution is shown in Figs. 2(a)–(f). In the uninsulated case of Fig. 2(a), the temperature at the edge of the slab changes from 21°C to 17°C within a distance of less than 0.3 m. As vertical insulation is added along the foundation wall, Fig. 2(b), the temperature near the slab edge increases by about 2°C. In addition, some isotherms 'slide' down along the vertical insulation resulting in warmer soil along the inner side of the foundation wall. An increase in the thermal resistance of the vertical insulation does not affect significantly the soil temperature field as illustrated in Fig. 2(c). However, when insulation is placed horizontally on the slab surface and vertically along the foundation wall, the soil temperature decreases substantially. This soil temperature decrease is significant just beneath the slab floor. The effect of increasing the vertical insulation length is best depicted by comparing Figs. 1(b) and (e). The soil continues to get warmer in the vicinity of the slab edge and the footing wall. When, the vertical insulation reaches the water table level, Fig. 2(f), the two zones of the ground separated by the vertical insulation becomes thermally decoupled. In particular, the soil temperature varies almost linearly with depth.

4. HEAT FLUX DISTRIBUTION

The heat flux $q(x)$, from a slab with vertical insulation can be obtained from the third-boundary condition of equation (1):

$$q(x) = \frac{2}{a}k_s H_f \sum_{n=1}^{+\infty}\left(C_n + \frac{(-1)^n}{\mu_n}T_i\right)\cos\mu_n x. \quad (12)$$

Figure 3 gives the heat flux distribution along a slab of width $2a = 10$ m. The temperature above the slab is $T_i = 21°\text{C}$ while the soil surface temperature is $T_s = 16°\text{C}$. At a depth $b = 5$ m, a water table exists with a constant temperature $T_w = 11°\text{C}$. Figure 3 indicates that by adding vertical insulation along the slab footing wall, heat losses are reduced from slab edges. The heat loss from the central area of the slab is not, however, affected by the added vertical insulation.

5. TOTAL SLAB HEAT LOSSES

The total heat loss, Q, from a vertically insulated slab is obtained by integrating the heat flux, $q(x)$, of equation (12) over $[-a, a]$:

$$Q = \frac{4}{a}k_s H_f \sum_{n=1}^{+\infty}\left(\frac{C_n}{\mu_n} + \frac{(-1)^n}{\mu_n^2}T_i\right). \quad (13)$$

Figure 4 shows the dependence of slab total heat loss, Q, on insulation depth for various values of H_v. As expected, when the depth of the vertical insulation

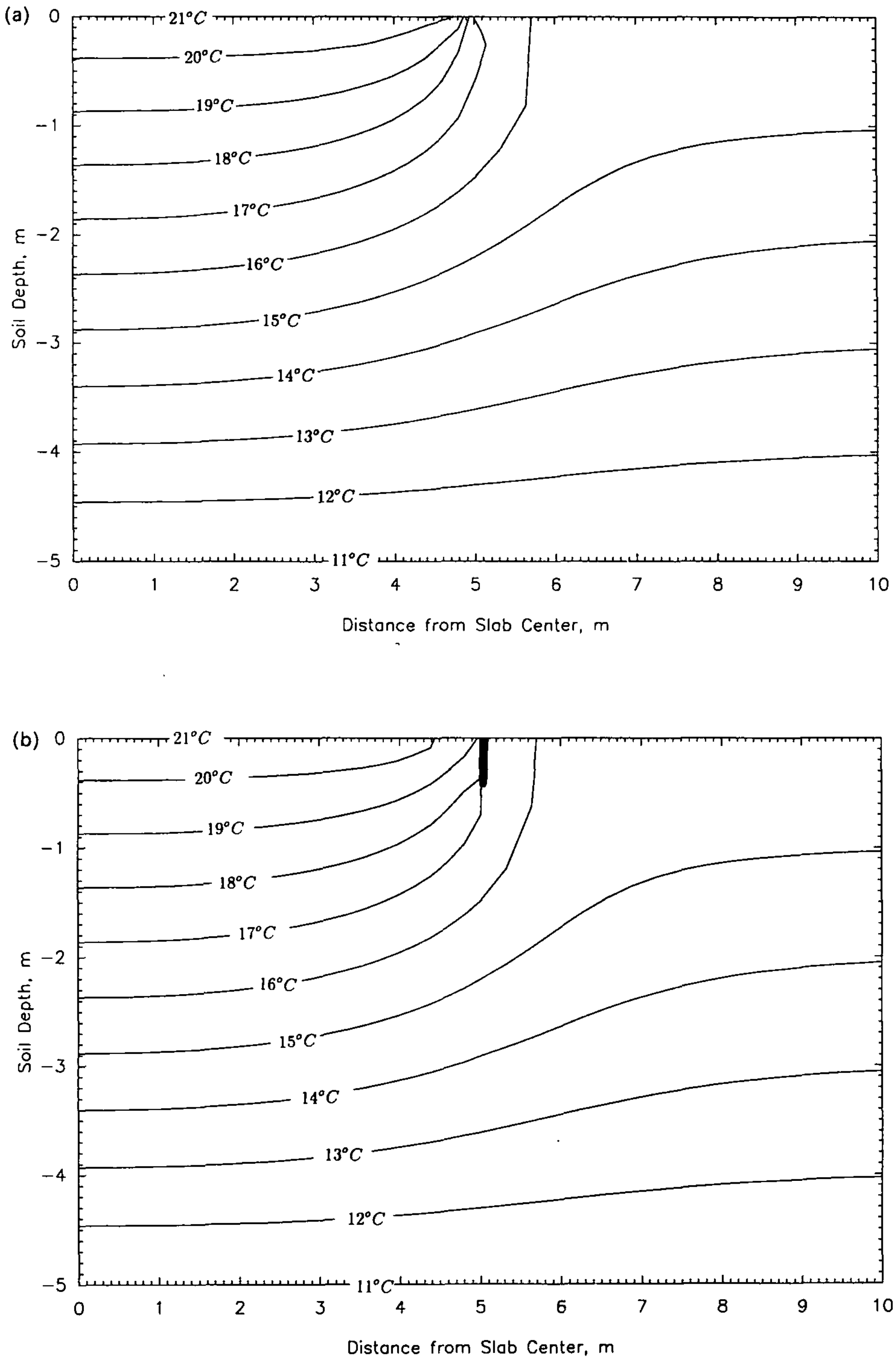
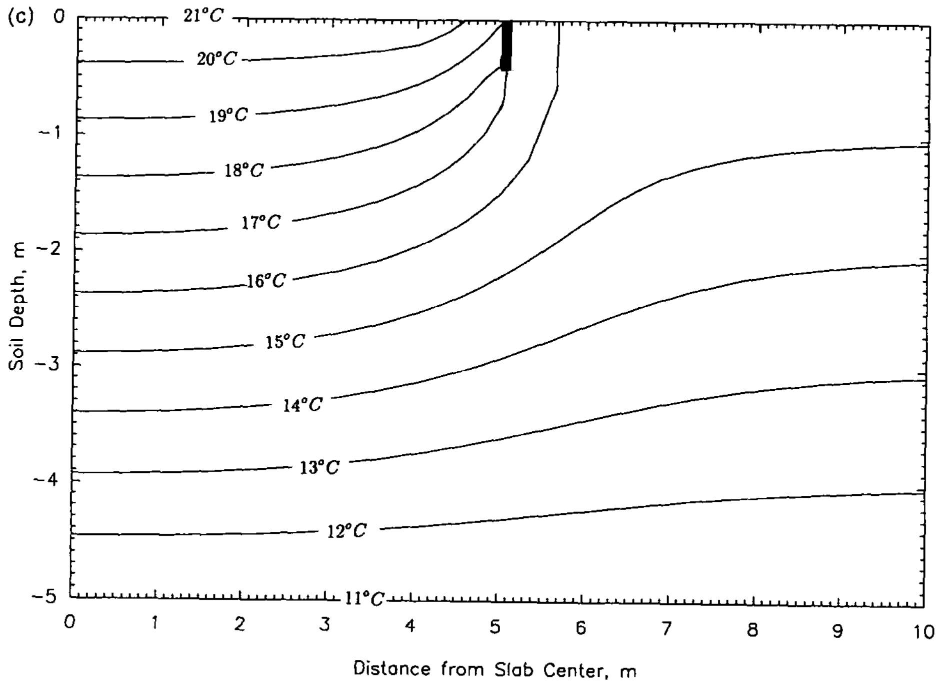


FIG. 2. Earth temperature isotherms beneath a slab-on-grade floor (a) $H_f = 4\ \mathrm{m}^{-1}$, $c = 0$ m; (b) $H_f = 4\ \mathrm{m}^{-1}$, $H_v = 0.1\ \mathrm{m}^{-1}$, $c = 0.5$ m; (c) $H_f = 4\ \mathrm{m}^{-1}$, $H_v = 0.05\ \mathrm{m}^{-1}$, $c = 0.5$ m; (d) $H_f = 0.5\ \mathrm{m}^{-1}$, $H_v = 0.1\ \mathrm{m}^{-1}$, $c = 0.5$ m; (e) $H_f = 4\ \mathrm{m}^{-1}$, $H_v = 0.1\ \mathrm{m}^{-1}$, $c = 2$ m; (f) $H_f = 4\ \mathrm{m}^{-1}$, $H_v = 0.1\ \mathrm{m}^{-1}$, $c = 5$ m.



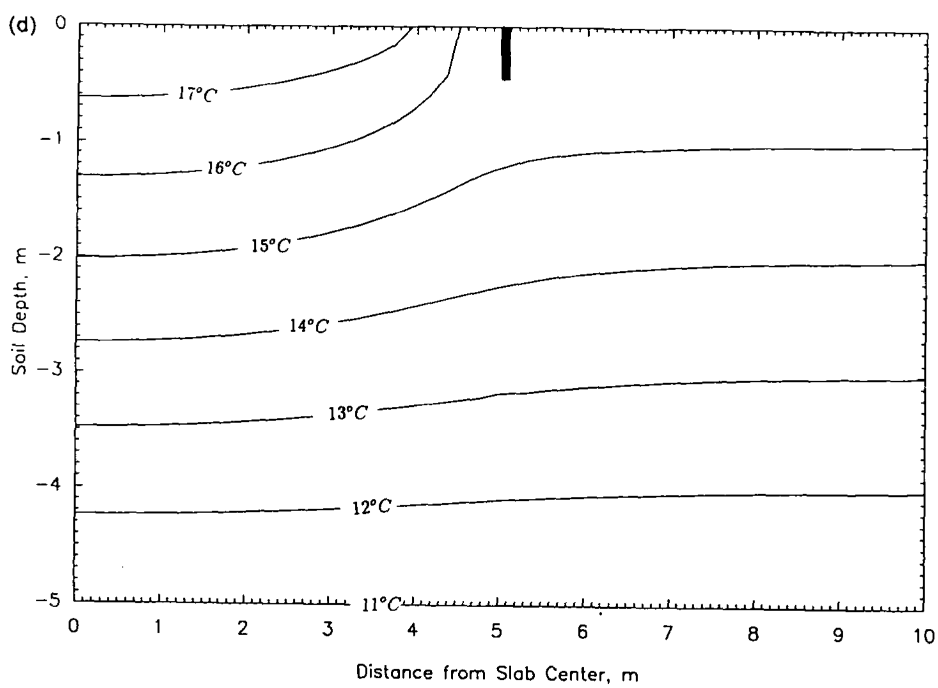
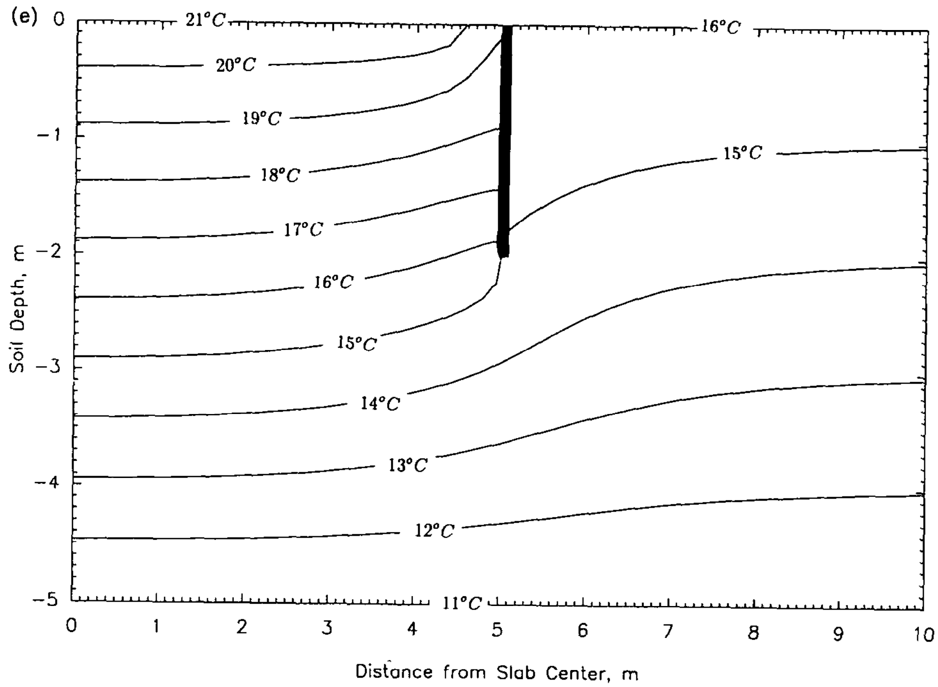


Fig. 2.—*Continued.*



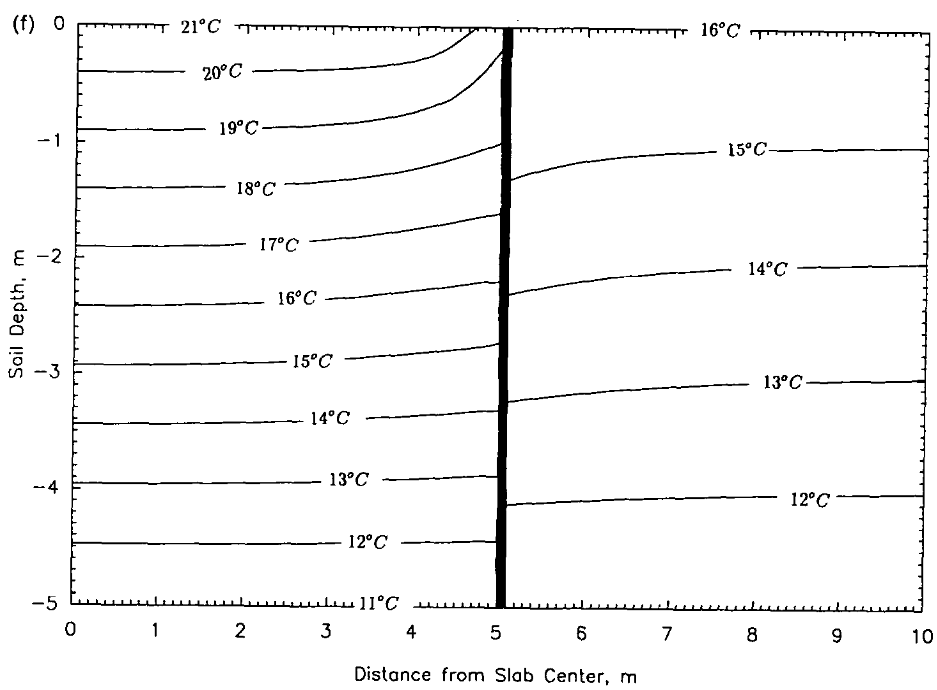


FIG. 2.—*Continued.*

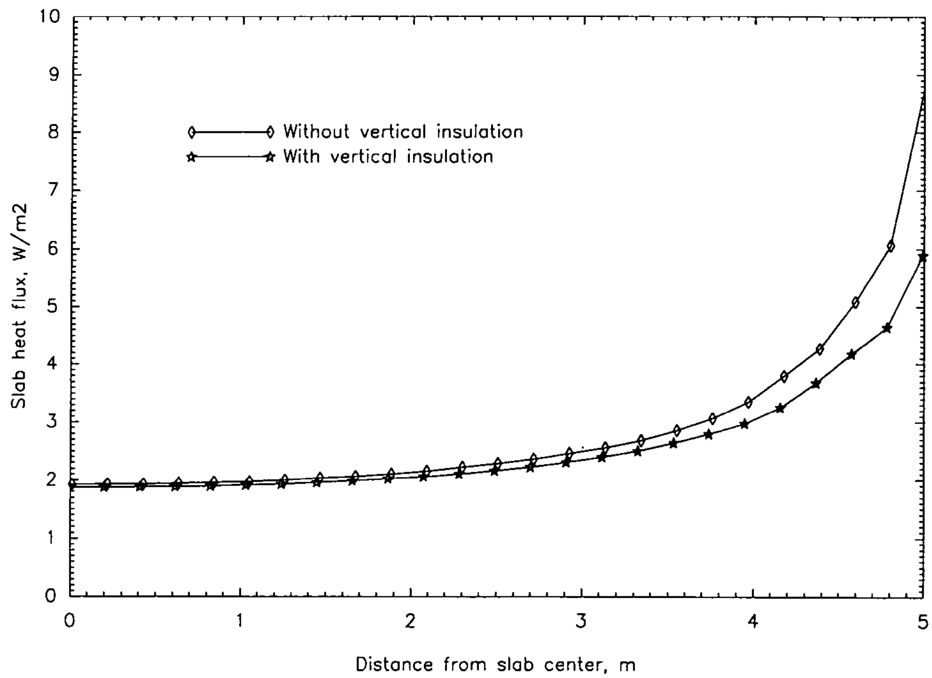


FIG. 3. Effect of vertical insulation on local heat flux distribution along an insulated slab-on-grade floor.

increases, slab heat loss decreases. This reduction follows the law of diminishing returns. Most of the reduction occurs in the first 0.5 m of the vertical insulation depth. Indeed, 0.5 m of vertical insulation with $H_v = 0.5\ m^{-1}$ placed along an uninsulated building foundation reduces the slab heat loss by 15%. If the depth of the same insulation is increased to 1 m, the slab heat loss will be reduced by 18%.

Figure 5 illustrates the effect of water table level on the total heat loss from a slab of width $2a = 10$ m and vertically insulated with $H_v = 0.5\ m^{-1}$. As the water table depth decreases, the slab loses significantly more heat. Indeed, the slab total heat loss doubles in magnitude when the water table depth decreases from 20 m to 2 m. As expected, Fig. 5 indicates that the effect of water table depth on slab heat loss decreases as the

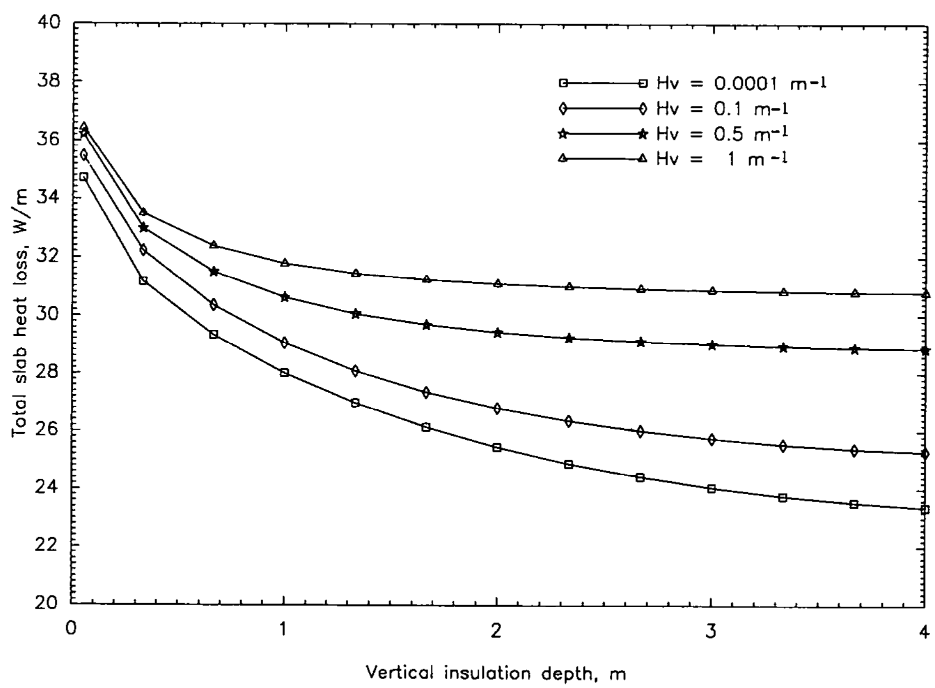


FIG. 4. Effect of vertical insulation and its depth on the total slab heat loss.

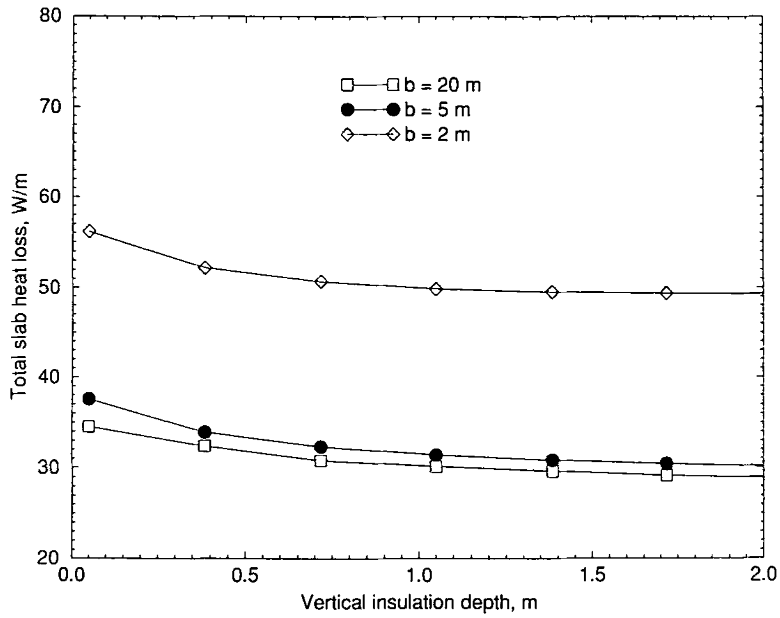


FIG. 5. Effect of water table level on the total slab heat loss.

water table depth increases. A change in the water table level from 20 m to 5 m results in less than 5% increase in the slab heat loss.

6. CONCLUSION

The steady-state heat transfer between soil and vertically insulated slabs in analyzed using the ITPE technique. It is shown that vertical edge insulation is effective in reducing the total heat losses from a building to the surrounding ground. As expected, the reduction in the foundation heat losses was found to closely depend on the depth and the U-value of the vertical insulation. However, it is found that soil temperature field and total slab heat loss are more sensitive to the vertical insulation depth than its thermal resistance. Finally, it is shown that the water table can affect significantly the slab heat losses.

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